TECHNICAL NOTE

ON THE CONVERGENCE OF MIEHLE'S ALGORITHM FOR THE EUCLIDEAN MULTIFACILITY LOCATION PROBLEM

J. B. ROSEN and G. L. XUE

University of Minnesota, Minneapolis, Minnesota

(Received October 1989; revision received April 1990; accepted August 1990)

For the Euclidean single facility location problem, E. Weiszfeld proposed a simple iterative algorithm in 1937. Later, it was proved by numerous authors that it is a convergent descent algorithm. W. Miehle extended Weiszfeld's algorithm to solve the Euclidean multifacility location problem. Then, L. M. Ostresh proved that Miehle's algorithm is a descent algorithm. Recently, F. Rado modified Miehle's algorithm and provided several sets of sufficient conditions for the modified algorithm to converge. He also indicated that the convergence of Miehle's algorithm was an open problem. In this paper, the relationship between Miehle's multifacility location algorithm and Weiszfeld's single facility location algorithm is analyzed. Counterexamples show that Miehle's algorithm may converge to nonoptimal points for well structured problems.

The problem of optimally locating one or several new service facilities with respect to the locations of a number of existing facilities (demand points) has generated a large literature (see the references listed in this paper). The Euclidean single facility location problem and the Euclidean multifacility location problem to be addressed here are among the models most often considered.

Let $a_1, a_2, \ldots, a_m$ be m noncollinear points (the locations of the existing facilities or the demand points) in R^d, the d-dimensional Euclidean space. Let $c_1, c_2, \ldots, c_m$ be m positive weights. Find a point x (the location of a new facility) that will

$$\underset{x \in R^d}{\text{minimize}}\ \phi(x) = \sum_{i=1}^{m} c_i \| x - a_i \|. \tag{1}$$

This is the so-called Euclidean single facility location problem.

Problem (1) has received a lot of attention in the literature. In 1937, in an effort to solve the following system of equations

$$\sum_{i=1}^{m} \frac{c_i(x - a_i)}{\| x - a_i \|} = 0, \tag{2}$$

Weiszfeld (1937) proposed the iteration formula

$$x_{k+1} = \left(\sum_{i=1}^{m} \frac{c_i}{\| x_k - a_i \|} \right)^{-1} \sum_{i=1}^{m} \frac{c_i a_i}{\| x_k - a_i \|}. \tag{3}$$

Then Kuhn (1974) proved that the algorithm is convergent. Several other authors also proved the convergence of the Weiszfeld algorithm independently (Wang 1975, Ostresh 1978, to cite a few).

If n new facilities are needed instead of *one*, we have a multifacility location problem to be stated formally. Let $a_1, a_2, \ldots, a_m$ be m noncollinear points in R^d. Let w_{ji}, $j = 1, 2, \ldots, n$, $i = 1, 2, \ldots, m$, and v_{jk}, $1 \leq j < k \leq n$ be nonnegative weights. Find a point $x = (x_1, \ldots, x_n)$ (the locations of new facilities) that will

$$\underset{x \in R^{n \times d}}{\text{minimize}}\ f(x) = \sum_{j=1}^{n} \sum_{i=1}^{m} w_{ji} \| x_j - a_i \| + \sum_{1 \leq j < k < n} v_{jk} \| x_j - x_k \|. \tag{4}$$

This is the so-called Euclidean multifacility location problem.

Subject classification: Facilities/equipment planning: multifacility location algorithm.
Area of review: DISTRIBUTION, TRANSPORTATION AND LOGISTICS.

0030-364x/92/4001-0188 $01.25

Let U be the set $\{(x_1, \ldots, x_n) \mid x_j \neq x_k \neq a_i \text{ for } j \neq k,\ i = 1, \ldots, m,\ 1 \leq j \leq n,\ 1 \leq k \leq n\}$. For any $x = (x_1, \ldots, x_n) \in U$, the gradient $\nabla f = (\nabla_1 f, \ldots, \nabla_n f)$ exists and is given by

$$(\nabla_j f)(x) = \sum_{i=1}^{m} w_{ji} \frac{x_j - a_i}{\| x_j - a_i \|} + \sum_{k \neq j} v_{jk} \frac{x_j - x_k}{\| x_j - x_k \|}, \quad j = 1, \ldots, n.$$

Given $x = (x_1, \ldots, x_n) \in U$, Miehle (1958) defined $x' = (x'_1, \ldots, x'_n)$ determined by the system of linear equations

$$\sum_{i=1}^{m} w_{ji} \frac{x'_j - a_i}{\| x_j - a_i \|} + \sum_{k \neq j} v_{jk} \frac{x'_j - x_k}{\| x_j - x_k \|} = 0, \quad j = 1, \ldots, n \quad (5)$$

as the next iteration point and claimed that for any initial point $x^0 \in U$, the sequence

$$x^0, x^1, \ldots, x^r, x^{r+1}, \ldots, \quad (6)$$

where $x^{r+1} = (x^r)'$, converges to a minimizer of f. Ostresh (1977) proved that algorithm (6) is a descent algorithm. Instead of solving the system of equations (5), each including a single unknown x'_j, Rado (1988) solved the following system of linear equations:

$$\sum_{i=1}^{m} w_{ji} \frac{x'_j - a_i}{\| x_j - a_i \|} + \sum_{k \neq j} v_{jk} \frac{x'_j - x'_k}{\| x_j - x_k \|} = 0, \quad j = 1, \ldots, n, \quad (5')$$

with unknowns $x'_1, \ldots, x'_n$, and proved that this algorithm converges to a minimizer of f for *well structured* problems. For a problem to be well structured it is sufficient, but not necessary, that the objective function f be strictly convex (see Rado for details). He also indicated that the convergence of Miehle's algorithm was an open problem.

In the next section, we will study the relationship between Miehle's algorithm for multifacility location problems and Weiszfeld's algorithm for single facility location problems, and construct counterexamples showing that Miehle's algorithm may converge to nonoptimal points for well structured problems.

1. MAIN RESULT

Consider the multifacility location problem (4) again. For any j, $1 \leq j \leq n$, if we fix the $n - 1$ points $\{x_k \mid k \neq j\}$, then the multifacility location problem (4) becomes the following single facility location problem:

$$\underset{y \in R^d}{\text{minimize}}\ \Psi(y) = \sum_{i=1}^{m} w_{ji} \| y - a_i \| + \sum_{k=1}^{j-1} v_{kj} \| y - x_k \| + \sum_{k=j+1}^{n} v_{jk} \| y - x_k \| \qquad P(x, j)$$

with $m + n - 1$ existing facilities $\{a_i \mid i = 1, \ldots, m\} \cup \{x_k \mid k = 1, \ldots, j-1, j+1, \ldots, n\}$.

If we take x_j as the initial value of y and apply one step of Weiszfeld's algorithm to problem P(x, j), the new value of y is exactly the x'_j defined by (5). This shows that one step of Miehle's algorithm to problem (4) is simply the application of one step of Weiszfeld's algorithm to problem P(x, j) for each $j \in \{1, \ldots, n\}$ independently. With this knowledge, we know that if Miehle's algorithm converges to a point $x = (x_1, \ldots, x_n)$, then for each $j \in \{1, \ldots, n\}$, x_j would be optimal for problem P(x, j). Also, Miehle's algorithm would get stuck at the point $x = (x_1, \ldots, x_n)$, if x_j is optimal for P(x, j) for each $j \in \{1, \ldots, n\}$, even when x is not optimal for problem (4). Therefore, it is easy to construct counterexamples for Miehle's algorithm to either stop at or converge to, a nonoptimal point.

Example 1. Consider an instance with $m = 4$ and $n = 2$. Let

$$a_1 = (-4, 1), \quad a_2 = (4, 1),$$

$$a_3 = (-4, -1), \quad a_4 = (4, -1),$$

$$w_{11} = w_{13} = 10, \quad w_{12} = w_{14} = 1,$$

$$w_{21} = w_{23} = 1, \quad w_{22} = w_{24} = 10,$$

$$v_{12} = \frac{-b + \sqrt{b^2 - 4c}}{2},$$

where

$$b = -\frac{50}{\sqrt{26}} - \frac{20}{\sqrt{5}} + \frac{6}{\sqrt{37}} + \frac{3}{\sqrt{10}},$$

$$c = -\frac{3200}{\sqrt{130}} + \frac{32}{\sqrt{370}}.$$

Let the initial point be $x^0 = (x_1^0, x_2^0)$, where $x_1^0 = (1, 0)$, $x_2^0 = (2, 0)$. It is clear that $x^0 \in U$. Applying one step of Miehle's algorithm, we get $x^1 = (x_1^1, x_2^1)$, where $x_1^1 = x_2^1 = (y, 0)$ with

$$y = \frac{2v_{12} - 80/\sqrt{26} + 8/\sqrt{10}}{v_{12} + 20/\sqrt{26} + 2/\sqrt{10}} \cong 1.5059103.$$

Since $v_{12} \cong 40.51604$ is greater than the sum of all the other weights, it follows from the optimality conditions for single facility location problems (Wang 1975, Xue 1989, Rosen and Xue 1991) that x_1^1 and x_2^1 are the optimal solutions for $P(x^1, 1)$ and $P(x^1, 2)$, respectively. Therefore Miehle's algorithm stops at this point (see Figure 1). However, x^1 is not the optimal solution because

$$f(0) = 44\sqrt{4^2 + 1}$$
$$< 22[\sqrt{(4+y)^2 + 1} + \sqrt{(4-y)^2 + 1}] = f(x^1).$$

In this example, v_{12} is a dominant weight. Therefore the optimal solution must satisfy $x_1 = x_2$. Once the two new facilities come near each other, the solution can be improved only by moving the two facilities *together* toward the origin. Since Miehle's algorithm moves each new facility *independently*, the algorithm stops at a nonoptimal point.

Example 2. If we change v_{12} in the above example to 100, the computational result from a simple Pascal program (see Table I) also shows that Miehle's algorithm converges to a point $\bar{x}$ = (0.61042520, 0; 0.61042520, 0), which is not optimal ($f(0) < f(\bar{x})$).

In this case, v_{12} is again a dominant weight and the optimal solution to (4) must satisfy $x_1 = x_2$. Therefore, the optimal solution of $P(x, j)$ must satisfy $y = x_j$, $j = 1, 2$. Once the two new facilities come near the point $\bar{x}$ (which is relatively far from the origin), rather than going toward the origin, they move closer and closer to each other and converge to $\bar{x}$, whose components are the optimal solutions for the corresponding single facility location problems.

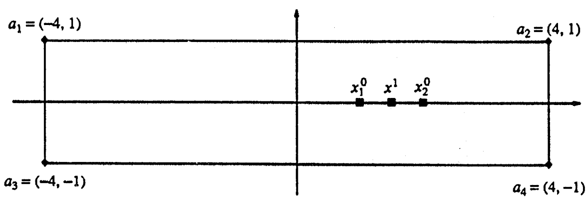

Figure 1. Miehle's algorithm stuck at a nonoptimal point x^1.

Table I
Convergence of Miehle's Algorithm to a Nonoptimal Point

Iteration	x_1	x_2
0	(1.00000000, 0)	(2.00000000, 0)
1	(1.76518900, 0)	(1.14113000, 0)
2	(1.05474700, 0)	(0.79469850, 0)
3	(0.75482910, 0)	(0.66854230, 0)
4	(0.65436630, 0)	(0.62796160, 0)
5	(0.62351070, 0)	(0.61563490, 0)
6	(0.61429640, 0)	(0.61196540, 0)
7	(0.61156820, 0)	(0.61087980, 0)
8	(0.61076250, 0)	(0.61055930, 0)
9	(0.61052470, 0)	(0.61046470, 0)
10	(0.61045440, 0)	(0.61043670, 0)
11	(0.61043370, 0)	(0.61042850, 0)
12	(0.61042760, 0)	(0.61042600, 0)
13	(0.61042580, 0)	(0.61042540, 0)
14	(0.61042520, 0)	(0.61042520, 0)

In both examples, the objective functions are strictly convex. Therefore, the problems are well structured. These two counterexamples adequately demonstrate that Miehle's algorithm for the Euclidean multifacility location problem can easily fail to converge to the optimal solution, even though the problem is strictly convex.

To conclude, note that if Miehle's algorithm stops at a point $x = (x_1, \ldots, x_n)$, then x_j is the optimal solution for the single facility location problem $P(x, j)$, $j = 1, \ldots, n$. However, x may be an optimal solution to the multifacility location problem.

ACKNOWLEDGMENT

We are grateful to the referee for constructive and helpful comments on the first version of this paper. This research was supported in part by Air Force Office of Scientific Research grant AFOSR-87-0127, National Science Foundation grant DCR-8420935, and a University of Minnesota Graduate School Doctoral Dissertation Fellowship.

REFERENCES

KUHN, H. W. 1974. Steiner's Problem Revisited. In *Studies in Optimization*, G. B. Dantzig and B. C. Eaves (eds.). The Mathematical Association of America, 52–70.

MIEHLE, W. 1958. Link Length Minimization in Networks. *Opns. Res.* **6**, 232–243.

OSTRESH, L. M. 1977. The Multifacility Location

Problem: Applications and Descent Theorems. *J. Region. Sci.* **17,** 409–419.

OSTRESH, L. M. 1978. On the Convergence of a Class of Iterative Methods for Solving the Weber Location Problem. *Opns. Res.* **26,** 597–609.

RADO, F. 1988. The Euclidean Multifacility Location Problem. *Opns. Res.* **36,** 485–492.

ROSEN, J. B., AND G. L. XUE. 1991. A Computational Comparison of Two Algorithms for the Euclidean Single Facility Location Problem. *ORSA J. Comput.* **4,** 207–212.

WANG, C. Y. 1975. On the Convergence and Rate of Convergence of an Iterative Algorithm for the Plant Location Problem. *Qufu Shiyun Xuebao* **2,** 14–25.

WEISZFELD, E. 1987. Sur le Point par Lequel le Somme des Distances de n Points Donnes est Minimum. *Tohoku Math. J.* **43,** 355–386.

XUE, G. L. 1989. A Globally and Quadratically Convergent Algorithm for min $\sum_{i=1}^{m} c_i \| x - a_i \|$ Type Plant Location Problem. *Acta Math. Appl. Sin.* **12,** 65–72.