ON THE CONVERGENCE OF A HYPERBOLOID APPROXIMATION PROCEDURE FOR THE PERTURBED EUCLIDEAN MULTIFACILITY LOCATION PROBLEM

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For the Euclidean single facility location problem, E. Weiszfeld proposed a simple closed-form iterative algorithm in 1937. Later, numerous authors proved that it is a convergent descent algorithm. In 1973, J. Oster, J. White and W. Wierwille extended Weiszfeld’s idea and proposed a Hyperboloid Approximation Procedure (HAP) for solving the Euclidean multifacility location problem. They believed, based on considerable computational experience, that the HAP always converges. In 1977, Ostresh proved that the HAP is a descent algorithm under certain conditions. In 1981, Morris proved that a variant of the HAP always converges. However, no convergence proof for the original HAP has ever been given. In this paper, we prove that the HAP is a descent algorithm and that it always converges to the minimizer of the objective function from any initial point.

The Euclidean multifacility location (EMFL) problem is one of locating new facilities in the Euclidean space with respect to existing facilities, the locations of which are known. The problem consists of finding locations of new facilities that will minimize a total cost function, which consists of a sum of costs directly proportional to the Euclidean distances between the new facilities, and costs directly proportional to the Euclidean distances between new and existing facilities. Mathematically, the problem can be stated as follows.

Let $a_1, a_2, \ldots, a_m$ be $m$ points in $\mathbb{R}^d$, the $d$-dimensional Euclidean space. Let $w_ij$, $j = 1, 2, \ldots, n_i$, $i = 1, 2, \ldots, m$, and $v_ijk$, $1 \leq j < k \leq n$ be given nonnegative numbers. For convenience, $v_ijk$ is assumed to be $v_jk$ whenever $j > k$ and $v_jj$ is assumed to be 0 for all $j$. Also, all vectors are assumed to be row vectors in this paper. The problem is to find a point $x = (x_1, \ldots, x_n) \in \mathbb{R}^{md}$ that will minimize

$$f(x) = \sum_{j=1}^{n} \sum_{i=1}^{m} w_ij \|x_j - a_i\|$$

$$+ \sum_{1 \leq i < j \leq n} v_ijk \|x_j - x_i\|,$$

(1)

where $\|\cdot\|$ is the Euclidean norm.

In the EMFL problem, $a_1, a_2, \ldots, a_m$ represent the location of $m$ existing facilities; $x_1, x_2, \ldots, x_n$ represent the locations of $n$ new facilities; the objective function $f(x)$ is the sum of weighted Euclidean distances from each new facility to each existing facility and those between each pair of new facilities; the objective is to find optimal locations for the new facilities, i.e., to minimize $f(x)$.

When $n = 1$, the EMFL problem becomes the Euclidean single facility location problem (ESFL). When the $m$ existing facilities are on a line, there is always an optimal solution which coincides with one of the existing facilities. Therefore, it is easy to solve. As a consequence, the $m$ existing facilities in ESFL are assumed to be noncolinear in the literature. For this problem, Weiszfeld (1937) gave a simple closed-form iterative algorithm. Later, numerous authors (Kuhn 1973, Wang et al. 1975, Ostresh 1977) proved that the algorithm converges globally and, under certain conditions, linearly. Brimberg (1989) also proved that global convergence of the original Weiszfeld algorithm is guaranteed for the $l_p$ distance when $p \in [1, 2]$, under the proviso that none of the iterates coincide with one of the existing facilities. There has been a large literature on Euclidean facility location problems. For more details and recent developments, readers are referred to the new book by Love, Morris and Wesolowsky (1988) and the papers by Chandrasekaran and Tamir (1989), Xue (1987, 1991), Rosen and Xue (1991) and Xue and Rosen (1992).

The objective functions of ESFL and EMFL are continuous and convex, but not differentiable at points where two or more facilities coincide. Nondifferentiability often presents difficulties in solving the
problem. For example, when the Weiszfeld algorithm is used to solve the ESFL problem, Wang et al. reports that convergence could be very slow when the minimizer coincides with one of the existing facilities. Also, when one of the Weiszfeld iterate points coincides with one of the existing facilities, a small step along the steepest descent direction has to be taken in order to further reduce the objective function value if the current point is not a minimizer. Rosen and Xue (1992) proved that the Miehle algorithm for the EMFL problem may converge to or stop at a nonoptimal point. In both counterexamples, the failure of the Miehle algorithm is caused by nondifferentiability.

For the ESFL problem, the objective function is nondifferentiable only at a finite number of points (where the new facility coincides with one of the existing facilities). If we take a small step along the steepest descent direction from the location of the existing facility which has the minimum objective function value among all the existing facilities, nondifferentiability can be avoided completely for further iterations. Therefore, the ESFL problem is essentially a smooth problem. For the EMFL problem, the objective function is nondifferentiable on one or more linear manifolds (where some facilities coincide). This makes the EMFL problem more difficult because nondifferentiability cannot be avoided easily. Despite this difficulty, there have been many approaches to solve the EMFL problem. Miehle (1958) was the first to propose an extension of the Weiszfeld algorithm for ESFL to solve the EMFL problem. Ostresh (1977) proved that Miehle’s algorithm is a descent one. However, as pointed out in a recent paper of Rosen and Xue (1992), Miehle’s algorithm may converge to or stop at a nonoptimal point. To avoid nondifferentiability in the EMFL problem, Eyster, White and Wierwille (1973) introduced a small positive number $\epsilon$ to the original problem, getting the following smooth perturbed objective function

$$ f(x) = \sum_{j=1}^{n} \sum_{i=1}^{m} w_{ji} \sqrt{\|x_j - a_i\|^2 + \epsilon} + \sum_{1 \leq j < k \leq n} u_{jk} \sqrt{\|x_j - x_k\|^2 + \epsilon}. $$

A minimum of $f(x)$ is called an $\epsilon$-optimal solution to the EMFL problem (1). Generalizing Weiszfeld’s algorithm and Miehle’s idea to minimize $f(x)$, they proposed a Hyperboloid Approximation Procedure (HAP) for solving EMFL. They could not prove convergence for the HAP but believed, based on considerable computational experience, that HAP always converges. Ostresh (1977) made a first step in proving the convergence of the HAP. He showed that there exists a positive number $\delta > 0$ such that the HAP is a descent algorithm whenever $\epsilon < \delta$. However it was not clear how small $\delta$ should be. In 1981, Morris proved global convergence for a variant of the HAP. In 1988, Rado proposed another extension of Weiszfeld’s algorithm and proved convergence under certain conditions. The convergence of the original HAP still remained open.

In this paper, we first simplify the problem by showing that an $\epsilon$-optimal solution of a general EMFL problem can be obtained from an $\epsilon$-optimal solution of a corresponding fully chained EMFL problem. We then prove that the HAP always converges for the fully chained problem, thus settling an open problem.

The rest of this paper is organized as follows. In Section 1, we describe the Miehle transformation and the HAP. In Section 2, we prove some preliminary theorems and the descent property of the HAP. In Section 3, we prove the global convergence of the HAP. In Section 4, we prove the global convergence of the Gauss-Seidel HAP.

1. THE HYPERBOLOID APPROXIMATION PROCEDURE

Since $f_i(x)$ is convex and continuously differentiable, a point $x$ is a global minimizer of $f_i(x)$ if and only if the gradient of $f_i(x)$ is zero at that point. Suppose that we have a current approximation $x = (x_1, x_2, \ldots, x_n)$ to the minimizer of $f_i(x)$ and want to further improve the location of the $j$th new facility. The gradient of $f_i$ with respect to the $j$th new facility $x_j$ is

$$ \nabla_j f_i(x) = \sum_{i=1}^{m} w_{ji} \frac{x_j - a_i}{\sqrt{\|x_j - a_i\|^2 + \epsilon}} + \sum_{k \neq j} u_{jk} \frac{x_j - x_k}{\sqrt{\|x_j - x_k\|^2 + \epsilon}}. $$

As in Weiszfeld (1937) and Miehle (1957), we may get an improved location $x_j^+$ of the $j$th new facility with respect to the existing facilities and the other new facilities by solving the system of linear equations

$$ \sum_{i=1}^{m} w_{ji} \frac{x_j^+ - a_i}{\sqrt{\|x_j^+ - a_i\|^2 + \epsilon}} + \sum_{k \neq j} u_{jk} \frac{x_j^+ - x_k}{\sqrt{\|x_j^+ - x_k\|^2 + \epsilon}} = 0 $$

for $x_j^+$. This gives

$$ x_j^+ = T_j(x_1, x_2, \ldots, x_n), $$

where $T_j: \mathbb{R}^{m+d} \to \mathbb{R}^d$, $j = 1, 2, \ldots, n$, is the Miehle
transformation (Miehle 1957, Ostresh 1977) defined by
\[
T_j(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{m} \frac{w_{ji} a_i}{\sqrt{\|x_j - a_i\|^2 + \epsilon}} + \sum_{k=1}^{m} \frac{v_{jk} x_k}{\sqrt{\|x_j - x_k\|^2 + \epsilon}}.
\]
(5)
This is essentially one step of Weiszfeld’s algorithm applied to the function \(f(x)\) when treated as a function of \(x_j\).

Applying (4) simultaneously to all \(j\) leads to the following Hyperboloid Approximation Procedure proposed by Eyster, White and Wierwille.

**Algorithm HAP**

1. **STEP.0.** Choose any initial point \(x^0 \in R^{n \times d}\). Set \(s := 0\) and go to Step_1.

2. **STEP.1.** For \(j := 1, 2, \ldots, n\) do \(x_j^{s+1} = T_j(x_1^s, x_2^s, \ldots, x_n^s)\).

3. **STEP.2.** If \(x^{s+1} = x^s\), stop; otherwise, replace \(s\) with \(s + 1\) and go to Step_1.

Note that the HAP is a closed-form iteration formula. It can be implemented in parallel very easily because the \(s + 1\)st approximation of the new facilities can be computed simultaneously from the existing facilities and the \(s\)th approximation of the new facilities.

Although the algorithm allows choosing an arbitrary initial point, it is preferred that the initial locations of the new facilities all lie in the convex hull of the locations of the existing facilities. One possible choice is the following center of gravity rule which is justified by Corollary 1. Let \(x^0\) be given by
\[
x_j^0 = \begin{cases} 
\left( \sum_{i=1}^{m} w_{ji} \right)^{-1} \sum_{i=1}^{m} w_{ji} a_i & \text{if } \sum_{i=1}^{m} w_{ji} > 0 \\
\left( \sum_{i=1}^{m} a_i \right) / m & \text{otherwise } j = 1, 2, \ldots, n.
\end{cases}
\]

The stopping rule in Step_2 of the algorithm is too restricted. Using this rule, the algorithm will almost always generate an infinite sequence. In actual implementation, this stopping test can be replaced by some more effective stopping rules. Interested readers are referred to the paper by Love and Dowling (1989) for a rational stopping rule which terminates the algorithm when the suboptimality of the current solution in EMFL is within a given tolerance.

### 2. PRELIMINARY THEOREMS AND DESCENT PROPERTIES

A new facility \(x_j\) and an existing facility \(a_i\) are said to have an exchange whenever \(w_{ji} > 0\). Two new facilities \(x_j\) and \(x_k\) are said to have an exchange whenever \(v_{jk} > 0\). A new facility \(x_j\) is said to be chained if there exist \(j_1, j_2, \ldots, j_t \in \{1, 2, \ldots, n\}\) and \(i_0 \in \{1, 2, \ldots, m\}\) such that \(v_{j_1i_0} \cdots v_{j_{t-1}i_0} w_{j_ti_0} \neq 0\). A variable \(x_j\) which is not chained is called a free variable. Let \(F\) and \(C\) be the index sets for free variables and chained variables, respectively. We can rewrite \(f(x)\) as
\[
f(x) = \sum_{j \in F} \sum_{i=1}^{m} w_{ji} \sqrt{\|x_j - a_i\|^2 + \epsilon} + \sum_{j \in C} \sum_{i=1}^{m} v_{jk} \sqrt{\|x_j - x_k\|^2 + \epsilon}.
\]
(6)
Due to the nonnegativity of the \(v_{jk}\)'s the function value in (6) will not increase for any \(x \in R^{n \times d}\) and \(x_{\text{free}} \in R^d\), if we change all the values of the free variables in \(x\) to \(x_{\text{free}}\). Furthermore, the first two terms in (6) make up another EMFL problem with no free variables. For this reason, with no loss of generality, we will assume in the rest of this paper that there is no free variable in the EMFL problem or, in terms of Francis and Cabot (1972), that the problem is fully chained.

With this assumption, \(f(x)\) is a strictly convex function for \(\epsilon > 0\) and has all orders of derivatives (Love, Morris and Wesolowsky, p. 87). In addition, we have the following properties.

**Proposition 1.** \(\lim_{\|x\| \to \infty} f(x) = +\infty\). Therefore, the EMFL problem has a unique \(\epsilon\)-optimal solution for each \(\epsilon > 0\).

**Proof.** If \(\|x\| \to \infty\), then \(\|x_{i_0}\| \to \infty\) for some \(i_0 \in \{1, 2, \ldots, n\}\). Since EMFL is fully chained, there exist \(j_1, j_2, \ldots, j_t \in \{1, 2, \ldots, n\}\) and \(i_0 \in \{1, 2, \ldots, m\}\) such that \(v_{j_1i_0} \cdots v_{j_{t-1}i_0} w_{j_ti_0} \neq 0\). As a consequence, we have
\[
\lim_{\|x\| \to \infty} \sum_{i=1}^{m} v_{jk} \sqrt{\|x_j - x_{k+1}\|^2 + \epsilon} + w_{j_ti_0} \sqrt{\|x_j - a_{i_0}\|^2 + \epsilon} = +\infty.
\]
(7)
Therefore \(\lim_{\|x\| \to \infty} f(x) = +\infty\). This, together with the continuity of \(f(x)\), guarantees the existence of a minimizer of \(f(x)\). Since \(f(x)\) is strictly convex, the minimizer is unique.
Proposition 2. For any \( j \in \{1, 2, \ldots, n\} \), \( T(x_1, x_2, \ldots, x_n) \) is in the convex hull of the points \( a_1, a_2, \ldots, a_m \) and \( x_1, x_2, \ldots, x_n \).

Proof. It follows from (5) that \( T(x_1, x_2, \ldots, x_n) \) is a convex combination of the points \( a_1, a_2, \ldots, a_m \) and \( x_1, x_2, \ldots, x_n \).

Theorem 1. Let \( x = (x_1, x_2, \ldots, x_n) \) be any point in \( \mathbb{R}^{n \times d} \). Let \( y = (y_1, y_2, \ldots, y_n) \) be the point generated by
\[
y_j = T(x_1, x_2, \ldots, x_n), \quad j = 1, 2, \ldots, n.
\] (8)
Then the following descent property for the HAP holds as long as \( \epsilon > 0 \):
\[
2f(x) - 2f(y) \geq \sum_{j=1}^{n} \sum_{i=1}^{m} w_{ji} \frac{\partial f}{\partial x_j} - \sum_{j=1}^{n} \sum_{i=1}^{m} w_{ji} \left( \frac{y_j - x_j}{\sqrt{\|y_j - x_j\|^2 + \epsilon}} + \frac{y_j - x_j}{\sqrt{\|y_j - x_j\|^2 + \epsilon}} \right) + \sum_{j=1}^{n} \sum_{i=1}^{m} u_{jk} \frac{y_k - x_k}{\sqrt{\|y_k - x_k\|^2 + \epsilon}} + \sum_{j=1}^{n} \sum_{i=1}^{m} u_{jk} \frac{x_j - x_k}{\sqrt{\|x_j - x_k\|^2 + \epsilon}}.
\] (9)
where \( \partial f = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right) \).

Proof. Since \( y_j = T(x_1, x_2, \ldots, x_n) \), it follows from (3) that
\[
\sum_{i=1}^{m} w_{ji} \frac{y_j - a_i}{\sqrt{\|y_j - a_i\|^2 + \epsilon}} + \sum_{k \neq j} u_{jk} \frac{y_j - x_k}{\sqrt{\|y_j - x_k\|^2 + \epsilon}} = 0.
\] (10)
Multiplying both sides of (10) by \( (y_j - x_j)^T \) and regrouping terms, we get
\[
\sum_{i=1}^{m} w_{ji} \frac{y_j - a_i}{\sqrt{\|y_j - a_i\|^2 + \epsilon}} \frac{y_j - x_j}{\sqrt{\|y_j - x_j\|^2 + \epsilon}} + \sum_{k \neq j} u_{jk} \frac{y_j - x_k}{\sqrt{\|y_j - x_k\|^2 + \epsilon}} + \sum_{k \neq j} u_{jk} \frac{x_j - x_k}{\sqrt{\|x_j - x_k\|^2 + \epsilon}} = \frac{\nabla f(x)(y_j - x_j)^T}{\sqrt{\|y_j - x_j\|^2 + \epsilon}}.
\] (11)
By definition,
\[
y_j = x_j - \nabla f(x) \left( \sum_{i=1}^{m} \frac{w_{ji}}{\sqrt{\|y_j - a_i\|^2 + \epsilon}} + \sum_{k \neq j} \frac{u_{jk}}{\sqrt{\|y_j - x_k\|^2 + \epsilon}} \right)^{-1}.
\] (12)
Therefore
\[
-\nabla f(x)(y_j - x_j)^T = \frac{\nabla f(x)(y_j - x_j)^T}{\sqrt{\|y_j - x_j\|^2 + \epsilon}} + \sum_{i=1}^{m} \frac{w_{ji}}{\sqrt{\|y_j - a_i\|^2 + \epsilon}} + \sum_{k \neq j} \frac{u_{jk}}{\sqrt{\|y_j - x_k\|^2 + \epsilon}}.
\] (13)
Combining (11) and (13) we get
\[
\sum_{i=1}^{m} w_{ji} \frac{\|y_j - a_i\|^2}{\sqrt{\|y_j - x_j\|^2 + \epsilon}} + \sum_{k \neq j} u_{jk} \frac{\|y_j - x_k\|^2}{\sqrt{\|y_j - x_j\|^2 + \epsilon}} + \frac{\|y_j - x_k\|^2}{\sqrt{\|x_j - x_k\|^2 + \epsilon}} = \sum_{i=1}^{m} w_{ji} \frac{\|x_j - a_i\|^2}{\sqrt{\|x_j - x_j\|^2 + \epsilon}} + \frac{\|x_j - x_k\|^2}{\sqrt{\|x_j - x_k\|^2 + \epsilon}} + \frac{\|x_j - x_k\|^2}{\sqrt{\|x_j - x_k\|^2 + \epsilon}}.
\] (14)
Summing (14) over \( j \) (note that (14) is true for all \( j \)), we get
\[
\sum_{i=1}^{m} \sum_{j=1}^{n} w_{ji} \frac{\|x_j - a_i\|^2}{\sqrt{\|x_j - x_j\|^2 + \epsilon}} + \sum_{k \neq j} \sum_{i=1}^{m} \frac{y_k - x_k}{\sqrt{\|y_k - x_k\|^2 + \epsilon}} + \frac{y_k - x_k}{\sqrt{\|y_k - x_k\|^2 + \epsilon}} = \sum_{i=1}^{m} \sum_{j=1}^{n} w_{ji} \frac{\|x_j - a_i\|^2}{\sqrt{\|x_j - x_j\|^2 + \epsilon}} + \frac{\|x_j - x_k\|^2}{\sqrt{\|x_j - x_k\|^2 + \epsilon}} + \frac{\|x_j - x_k\|^2}{\sqrt{\|x_j - x_k\|^2 + \epsilon}}.
\] (15)
where \( \partial f = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right) \).

Combining the inequality
\[
\|y_j - x_k\|^2 + \|y_j - x_j\|^2 + \|y_k - x_j\|^2 + \|y_k - x_k\|^2 \geq \|x_j - x_k\|^2 + \|y_j - y_k\|^2.
\] (16)
with (15), we get
\[
\sum_{i=1}^{m} \sum_{j=1}^{n} w_{ji} \frac{\|y_j - a_i\|^2 + \|y_j - x_j\|^2}{\sqrt{\|x_j - a_i\|^2 + \epsilon}} + \sum_{k \neq j} \sum_{i=1}^{m} \frac{y_k - x_k}{\sqrt{\|y_k - x_k\|^2 + \epsilon}} + \frac{y_k - x_k}{\sqrt{\|y_k - x_k\|^2 + \epsilon}} \leq \sum_{i=1}^{m} \sum_{j=1}^{n} w_{ji} \frac{\|x_j - a_i\|^2}{\sqrt{\|x_j - x_j\|^2 + \epsilon}} + \frac{\|x_j - x_k\|^2}{\sqrt{\|x_j - x_k\|^2 + \epsilon}} + \frac{\|x_j - x_k\|^2}{\sqrt{\|x_j - x_k\|^2 + \epsilon}}.
\] (17)
Adding
\[ \sum_{j=1}^{n} \sum_{i=1}^{m} w_{ij} \frac{\| y_j - a_i \|^2 + \epsilon}{\sqrt{\| x_j - a_i \|^2 + \epsilon}} \]
\[ + \sum_{j=1}^{n} \sum_{k=j}^{m} v_{jk} \frac{\| y_j - x_k \|^2 + \epsilon}{\sqrt{\| x_j - x_k \|^2 + \epsilon}} \]
to both sides of (17), we get
\[ \sum_{j=1}^{n} \sum_{i=1}^{m} w_{ij} \frac{\| y_j - a_i \|^2 + \epsilon + \| y_j - x_j \|^2}{\sqrt{\| x_j - a_i \|^2 + \epsilon}} \]
\[ + \sum_{j=1}^{n} \sum_{k=j}^{m} v_{jk} \frac{\| y_j - x_k \|^2 + \epsilon}{\sqrt{\| x_j - x_k \|^2 + \epsilon}} \leq f(x). \]
(18)

Following the ideas of Ostresh (1977) and Weiszfeld, we have
\[ \| y_j - a_i \|^2 + \epsilon \]
\[ = (\sqrt{\| y_j - a_i \|^2 + \epsilon} - \sqrt{\| x_j - a_i \|^2 + \epsilon})^2 \]
\[ + 2 \sqrt{\| y_j - a_i \|^2 + \epsilon} \cdot \sqrt{\| y_j - a_i \|^2 + \epsilon} \]
\[ + (\sqrt{\| x_j - a_i \|^2 + \epsilon} - \sqrt{\| x_j - a_i \|^2 + \epsilon})^2 \]
\[ \leq f(x). \]
(19)

\[ \| y_j - x_k \|^2 + \epsilon \]
\[ = (\sqrt{\| y_j - x_k \|^2 + \epsilon} - \sqrt{\| x_j - x_k \|^2 + \epsilon})^2 \]
\[ + 2 \sqrt{\| y_j - x_k \|^2 + \epsilon} \cdot \sqrt{\| y_j - x_k \|^2 + \epsilon} \]
\[ + (\sqrt{\| x_j - x_k \|^2 + \epsilon} - \sqrt{\| x_j - x_k \|^2 + \epsilon})^2 \]
\[ \leq f(x). \]
(20)

Substituting (19) and (20) into (18), we get
\[ \sum_{j=1}^{n} \sum_{i=1}^{m} w_{ij} \sqrt{\| x_j - a_i \|^2 + \epsilon} \]
\[ + \sum_{j=1}^{n} \sum_{k=j}^{m} v_{jk} \frac{\sqrt{\| y_j - x_k \|^2 + \epsilon} - \sqrt{\| x_j - x_k \|^2 + \epsilon}}{\sqrt{\| x_j - x_k \|^2 + \epsilon}} \]
\[ + 2f(y) - f(x) \leq f(x), \]
(21)
where \( \delta = (\sqrt{\| y_j - a_i \|^2 + \epsilon} - \sqrt{\| x_j - a_i \|^2 + \epsilon})^2 + \| y_j - x_k \|^2 + \epsilon. \)

This is equivalent to (9).

Ostresh (1977) proved that there exists a constant \( \delta > 0 \) such that the HAP is a descent algorithm whenever \( 0 < \epsilon \leq \delta \). Theorem 1 is an extension of Ostresh's descent property because it asserts that the HAP is a descent algorithm for all \( \epsilon > 0 \).

3. GLOBAL CONVERGENCE OF THE HAP

In this section, we will prove that the HAP is strictly decreasing at each step. Then we prove that the algorithm always converges from any initial point. We assume throughout this section that \( y = (y_1, \ldots, y_n) = (T_1(x), \ldots, T_n(x)) \).

**Lemma 1.** If there exist \( j_0 \in \{1, 2, \ldots, n\} \) and \( i_0 \in \{1, 2, \ldots, m\} \) such that
\[ w_{j_0 i_0} > 0, \quad \nabla_{j_0} f(x) \neq 0, \]
then
\[ f(y) \leq f(x) - \frac{w_{j_0 i_0}}{2\sqrt{\| x_{j_0} - a_{i_0} \|^2 + \epsilon}} \cdot \| y_{j_0} - x_{j_0} \|^2 \]
\[ < f(x). \]
(22)

**Proof.** From Theorem 1 we have
\[ f(y) \leq f(x) - \frac{w_{j_0 i_0}}{2\sqrt{\| x_{j_0} - a_{i_0} \|^2 + \epsilon}} \cdot \| y_{j_0} - x_{j_0} \|^2. \]

Combining (12) and (22) we get
\[ \frac{w_{j_0 i_0}}{2\sqrt{\| x_{j_0} - a_{i_0} \|^2 + \epsilon}} \cdot \| y_{j_0} - x_{j_0} \|^2 > 0. \]
(23)

This completes the proof.

**Lemma 2.** If there exist \( j_0, k_0 \in \{1, 2, \ldots, n\} \) such that \( u_{j_0 k_0} > 0, \nabla_{k_0} f(x) \neq 0 \) and \( \nabla_{j_0} f(x) = 0 \), then
\[ f(y) \leq f(x) - \frac{u_{j_0 k_0}}{2\sqrt{\| x_{j_0} - x_{k_0} \|^2 + \epsilon}} \cdot \| y_{j_0} - x_{j_0} \|^2 < f(x). \]

**Proof.** It follows from \( \nabla_{k_0} f(x) = 0 \) and the definition of the Miehle transformation that \( y_{k_0} = x_{k_0} \). If we fix \( x_{k_0} \) at its current value, treat \( x_{k_0} \) as an extra existing facility instead of a new facility, and treat \( u_{j_0 k_0} \) as the weight on the distance from the \( j \)-th new facility to this extra existing facility for \( j \neq k_0 \), then we can consider the current EMFL problem as a new EMFL problem with \( m + 1 \) existing facilities and \( n + 1 \) new facilities. Taking one step of the HAP algorithm on this new problem will result in exactly the same values for the \( j \)-th new facility for all \( j \neq k_0 \). Applying Lemma 1 to this new EMFL problem, we get the desired inequality.

**Theorem 2.** If \( \nabla f(x) \neq 0 \), then \( f(y) < f(x) \).

**Proof.** Since \( \nabla f(x) \neq 0 \), there exists \( j_0 \in \{1, 2, \ldots, n\} \) such that \( \nabla_{j_0} f(x) \neq 0 \). Since variable \( x_{j_0} \) is chained, there exist \( j_1, j_2, \ldots, j_l \in \{1, 2, \ldots, n\} \) and \( i_0 \in \{1, 2, \ldots, n\} \) such that \( u_{j_1 i_0} \leq u_{j_2 i_0} \leq \cdots \leq u_{j_l i_0} > 0 \).

Let \( r = \max \{ i | \nabla f(x) \neq 0, 0 \leq i \leq l \} \). If \( r = l \), it follows from Lemma 1 that \( f(y) < f(x) \). If \( r < l \), it follows from Lemma 2 that \( f(y) < f(x) \).

**Theorem 3.** From any initial point \( x^0 \in R^{n \times 4} \), the HAP either stops at the \( \epsilon \)-optimal solution of EMFL...
or generates an infinite sequence \( \{x^k\} \) converging to the \( \epsilon \)-optimal solution of the EMFL.

**Proof.** If the HAP stops at some iteration, then \( x^{s+1} = x^s \) for some integer \( s \). It follows from the definition of the algorithm that \( \nabla f_j(x^s) = 0 \), \( j = 1, 2, \ldots, n \). Therefore \( \nabla f_j(x) = 0 \) and \( x^s \) is the \( \epsilon \)-optimal solution of EMFL.

Now suppose that HAP generates an infinite sequence \( \{x^k\} \). It follows from Proposition 2 that \( \{x^k\} \) is bounded. Suppose that \( \{x^k\} \) does not converge to the \( \epsilon \)-optimal solution of EMFL, there would exist a subsequence \( \{x^{k'}\} \) that converges to a point \( \tilde{x} \), which is not the \( \epsilon \)-optimal solution of EMFL. Without loss of generality, we may assume that the subsequent \( \{x^{k'}\} \) converges to some point \( \tilde{x} \).

From the continuity of \( f_j(x) \) we have

\[
\lim_{r \to 0} f_j(x^s) = f_j(\tilde{x}), \quad \lim_{r \to 0} f_j(x^{s+1}) = f_j(\tilde{x}). \tag{24}
\]

Since \( f_j(x^s) \) is monotonically decreasing, (24) implies

\[
f_j(\tilde{x}) = \lim_{r \to 0} f_j(x^s) = \lim_{r \to 0} f_j(x^{s+1}) = f_j(\tilde{x}). \tag{25}
\]

It follows from the continuity of \( T(x) \) that

\[
\tilde{x} = \lim_{r \to 0} x^{s+1} = \lim_{r \to 0} T(x^s) = T(\tilde{x}). \tag{26}
\]

It follows from Theorem 2 that

\[
f_j(\tilde{x}) < f_j(\tilde{x}). \tag{27}
\]

This is in contradiction with (25) and the proof is complete.

**Corollary 1.** The unique \( \epsilon \)-optimal solution of the EMFL problem is in the convex hull of the existing facilities.

**Proof.** Start the HAP with any point in the convex hull of the existing facilities as the initial point. From Lemma 2, the sequence \( \{x^k\} \) is in the convex hull of the existing facilities. From Theorem 2, the unique \( \epsilon \)-optimal solution for the EMFL problem is either one of these points or the limit of this sequence. Therefore, it is in the convex hull of the existing facilities.

### 4. GLOBAL CONVERGENCE OF THE GAUSS-SEIDEL HAP

The algorithm described in Section 1 is a Jacobi-like algorithm because all the new facilities change locations at the same time, in the sense that \( x^{k+1}_j \) is a function of \( x_1^k, \ldots, x_{j-1}^k, x_j^k, \ldots, x_n^k \), for all \( j \). Ostresh (1977) suggests that an alternative, and potentially more efficient procedure is to use new information as it becomes available. This leads to the following Gauss-Seidel HAP.

**Algorithm GSHAP**

**STEP 0.** Choose any initial point \( x^0 \in R^{n \times d} \). Set \( s := 0 \) and go to Step 1.

**STEP 1.** For \( j := 1, 2, \ldots, n \) do \( x^{s+1} = T_j(x^s, x_1^s, \ldots, x_n^s), s = s + 1 \).

**STEP 2.** If \( x^s = x^{s\text{-opt}} \), stop; otherwise go to Step 1.

Morris (1981) proves that the Gauss-Seidel HAP is globally convergent by proving that the algorithmic map of the Gauss-Seidel HAP is closed. We will see that the global convergence of the Gauss-Seidel HAP also follows from Theorem 1. Theorem 1 states that a sufficient decrease in the objective function value is achieved at every iteration of the Jacobi HAP. A simplification of Theorem 1 guarantees a sufficient decrease in the objective function value when the HAP is applied in solving the ESFL problem. Therefore, a sufficient decrease in the objective function value is also achieved at every iteration of the Gauss-Seidel HAP (which consists of \( n \) iterations of the HAP applied to \( n \) ESFL problems). The global convergence of the algorithm then follows from this sufficient decrease property. These ideas are described in detail in the following theorems.

**Theorem 4.** Let \( x = (x_1, x_2, \ldots, x_n) \) be any point in \( R^{n \times d} \). Fix \( j \in [1, n] \). Let \( y = (y_1, y_2, \ldots, y_n) \) be a point such that \( y_j = T_j(x_1, x_2, \ldots, x_n) \) and \( y_k = x_k, k \neq j \). Then the following descent property holds as long as \( \epsilon > 0 \).

\[
2f_j(x) - 2f_j(y) \geq \|y_j - x_j\|^2 - \sum_{i=1}^{m} \frac{w_{ij}}{\sqrt{\|x_j - a_i\|^2 + \epsilon}} + \sum_{k \neq j} \frac{u_{jk}}{\sqrt{\|x_j - x_k\|^2 + \epsilon}}. \tag{28}
\]

**Proof.** Consider this problem as an ESFL with \( m + n - 1 \) existing facilities (the \( m \) \( a_i \)'s and the \( (n - 1) \) \( x_k \)'s with \( k \neq j \)) and one new facility \( x_j \). Applying Theorem 1 to this EMFL problem, we get

\[
2f_j(x) - 2f_j(y) \geq \sum_{i=1}^{m} \frac{w_{ij}}{\sqrt{\|x_j - a_i\|^2 + \epsilon}} + \sum_{k \neq j} \frac{u_{jk}}{\sqrt{\|x_j - x_k\|^2 + \epsilon}} + \epsilon.
\]
\[ \sum_{i=1}^{m} \frac{w_{ji}}{\sqrt{\|x_j - a_i\|^2 + \epsilon}} + \sum_{j \neq i} \frac{v_{jk}}{\sqrt{\|x_j - x_k\|^2 + \epsilon}} \]

where \( a = (\sqrt{\|y_j - a_i\|^2 + \epsilon} - \sqrt{\|x_j - a_i\|^2 + \epsilon})^2 + \|y_j - x_k\|^2 \), and

where \( b = (\sqrt{\|y_j - x_k\|^2 + \epsilon} - \sqrt{\|x_j - x_k\|^2 + \epsilon})^2 + \|y_j - x_k\|^2 \).

This completes the proof.

**Corollary 2.** Let \( \{x^k\} \) be the sequence generated by the GSHAP. Then \( \{f(x^k)\} \) is a nonincreasing sequence. Furthermore, there exists a positive constant \( c \) (which depends on \( x^k \)) such that

\[ 2f(x^{k+1}) - 2f(x^k) \geq c \|x^{k+1} - x^k\|^2, \quad k = 1, 2, 3, \ldots \quad (29) \]

\[ 2f(x^i) - 2f(x^{i+1}) \geq c \|x^{i+1} - x^i\|^2, \quad i = 1, 2, \ldots, n, \quad k = 1, 2, 3, \ldots \quad (30) \]

**Proof.** It follows from (28) that \( \{f(x^k)\} \) is a nonincreasing sequence. Now define

\[ c = \inf_{x \in \mathbb{R}^{m \times d}} \{f(x) \leq f(x^0)\}. \]

Since the problem is fully chained by assumption and the domain \( \{x \mid f(x) \leq f(x^0)\} \) is bounded, the above defined \( c \) is a positive number.

Since only one new facility (call it the moving facility) may change its location during one iteration of the GSHAP, we have \( \|x^{k+1} - x^k\| = \|x^j - T(x^{k+1})\| \), where \( j \) is the index of the moving new facility. It then follows from Theorem 4 and the definition of \( c \) that (29) holds. Now, (30) follows from (29) and the fact that

\[ \sum_{p=1}^{i} \|x^{k+p} - x^{k+p-1}\|^2 = \|x^{k+i} - x^k\|^2, \quad i = 1, 2, \ldots, n. \]

The proof is complete.

**Theorem 5.** From any initial point \( x^0 \in \mathbb{R}^{n \times d} \), the GSHAP either stops at the \( \epsilon \)-optimal solution of the EMFL or it generates an infinite sequence \( \{x^i\} \) converging to the \( \epsilon \)-optimal solution of the EMFL.

**Proof.** The GSHAP stops only at points where the gradient of the objective function is zero. Therefore, we need only to consider the latter case.

Suppose that the GSHAP generates an infinite sequence \( \{x^i\} \). It follows from Proposition 2 that \( \{x^i\} \) is bounded. Suppose that \( \{x^i\} \) does not converge to the \( \epsilon \)-optimal solution of the EMFL, there would exist a subsequence \( \{x^i\} \) and \( n \) points \( x^i (i = 1, 2, \ldots, n) \), such that \( r \) is always a multiple of \( n \) and \( \{x^{i+1}\} \) converges to point \( x^i (i = 1, 2, \ldots, n) \), none of which is the \( \epsilon \)-optimal solution of the EMFL.

Since \( \{f(x^k)\} \) is bounded from below, it follows from the continuity of \( f(x) \) and inequality (30) that all \( n \) points \( x^i (i = 1, 2, \ldots, n) \) are identical. Let us call this point \( x^\hat{i} \). Since \( x^\hat{i} \) is not the \( \epsilon \)-optimal solution of the EMFL, there exists an index \( j \) such that \( \nabla f(x^\hat{i}) \neq 0 \). It follows from (12) that \( x^i \neq T(x^\hat{i}) \). Let \( \delta = \frac{1}{2}\|x^i - T(x^\hat{i})\| \). Then it follows from the continuity of \( T \cdot \) that \( \|x^{i+1} - x^i\| \geq \delta \) for \( r_i \), large enough. This fact, together with inequality (29), contradicts the fact that \( \{f(x^i)\} \) is bounded from below. The contradiction also completes the proof.

To conclude, we have proved that the HAP for solving the perturbed EMFL solution is globally convergent. This resolves an open question in location theory. We have also provided an alternative proof of the global convergence of the Gauss-Seidel HAP for solving the EMFL problem.

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