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Steiner tree problem with minimum number of Steiner points and bounded edge-length

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Abstract

In this paper, we study the *Steiner tree problem with minimum number of Steiner points and bounded edge-length* (STP-MSPBEL), which asks for a tree interconnecting a given set of n terminal points and a minimum number of Steiner points such that the Euclidean length of each edge is no more than a given positive constant. This problem has applications in VLSI design, WDM optimal networks and wireless communications. We prove that this problem is NP-complete and present a polynomial time approximation algorithm whose worst-case performance ratio is 5.

Keywords: Algorithms; Approximation algorithms; Steiner minimum trees; VLSI design; WDM optimal networks; Wireless communications

1. Introduction

Given a set of n terminal points $P = \{p_1, p_2, \ldots, p_n\}$ in the two-dimensional Euclidean plane $\mathbb{R}^2$, and a positive constant R, the *Steiner tree problem with minimum number of Steiner points and bounded edge-length*, denoted by STP-MSPBEL for short, asks for a tree spanning a superset of P such that each edge in the tree has a length no more than R and the number of points other than those in P, called *Steiner points* [2, 3,5,6], is minimized.

In the classical Steiner tree problem, a Steiner point always has a degree of 3. In the STP-MSPBEL problem, however, degree-2 Steiner points are possible. For example, when $n = 2$ and $d(p_1, p_2)$, the Euclidean distance between p_1 and p_2, is larger than R, the optimal tree has $\lceil d(p_1, p_2)/R \rceil - 1$ Steiner points, each of which has a degree of 2.

The STP-MSPBEL problem has important applications in wavelength-division multiplexing (WDM) optimal network design. Suppose we need to connect n sites located at $p_1, p_2, \ldots, p_n$ using a WDM optimal network. Due to the limit in transmission power, signals can only travel a limited distance (say R) in order to guarantee correct transmission. If some of the inter-site distances are greater than R, we need to provide some amplifiers or receivers/transmitters at some locations so that there is a spanning tree spanning the n sites and the receivers/transmitters so that the length of every edge is no more than R. For more details on

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these applications, see [7,8]. The STP-MSPBEL problem also finds applications in VLSI design [1,4,10], and the evolutionary/phylogenetic tree constructions in computational biology [6].

In this paper, we show that the STP-MSPBEL problem is NP-complete and present a polynomial time approximation algorithm with a constant performance ratio.

The rest of this paper is organized as follows. In Section 2, we prove that the decision version of the STP-MSPBEL problem is NP-complete. In Section 3, we present a polynomial time approximation algorithm for solving STP-MSPBEL with a performance ratio of 5. We conclude this paper in Section 4.

2. NP-completeness of STP-MSPBEL

Throughout this paper, we will use $V(T)$ and $E(T)$ to denote the vertex set and edge set of a tree T. We will also use $|Y|$ to denote the cardinality of set Y. Furthermore, we will assume standard graph theoretic notations [11] unless otherwise specified.

In this section, we will prove that the decision version of the STP-MSPBEL problem is NP-complete. The NP-hardness of the problem is proved by a polynomial time reduction from the *Discrete Euclidean Steiner Minimum Tree* problem which is known to be NP-hard [3]. Let us formally define the two decision problems in the following.

Problem 1 (*Discrete Euclidean Steiner minimum tree*). Given a set X of integer-coordinate points in the Euclidean plane and a positive integer L, does there exist a set $Y \supseteq X$ of integer-coordinate points such that some spanning tree T for Y satisfies $l'(T) \leqslant L$?

Here $l'(T)$ stands for the *discrete* length of tree T, which is the sum of the *discrete* Euclidean lengths of its edges. Given two points x and y in the plane, the *discrete* length of the edge joining them is $d'(x,y) = \lceil d(x,y) \rceil$ (where $d(x,y)$ is the Euclidean distance between points x and y, and $\lceil \alpha \rceil$ is the least integer not less than α).

Problem 2 (*STP with minimum number of Steiner points and bounded edge-length*). Given a set P of n terminal points in the two-dimensional Euclidean plane $\mathbb{R}^2$, a positive constant R, and a non-negative integer B. The problem asks whether there exists a tree spanning a point set $Q \supseteq P$ such that each edge in the tree has a length no greater than R and the number of Steiner points (points in $Q \setminus P$) is less than or equal to B.

Theorem 3. *There is a polynomial time reduction from Problem* 1 *to Problem* 2.

Proof. Let I be an instance of Problem 1. We construct an instance I' of Problem 2 by letting $P = X$, $R = 1$ and $B = L - (|X| - 1)$. It is clear that this construction of I' from I takes polynomial time.

Let T' be a solution to I', i.e., T' is a tree spanning a superset Y of X such that $|Y \setminus X| \leqslant L - (|X| - 1)$ and such that the Euclidean length of each edge in T' is no more than 1. Since the Euclidean length of each edge in T' is no more than 1, the discrete length of each edge in T' is no more than 1. Therefore $l'(T') \leqslant |Y| - 1$ since there are $|Y| - 1$ edges in T'. However,

$$\begin{aligned} |Y| - 1 &= |Y \setminus X| + |X| - 1 \\ &\leqslant L - \big(|X| - 1\big) + |X| - 1 \\ &= L. \end{aligned} \tag{1}$$

Therefore T' is also a solution to I. To summarize, we have proved that *any solution to I' is also a solution to I*.

Now assume that T is a solution to I. Therefore T is a tree which spans a superset Y of X such that $l'(T) \leqslant L$. Note that the number of edges in T is $|Y| - 1$.

For each edge e in T, we insert $l'(e) - 1$ equally spaced degree-2 Steiner points to divide edge e into $l'(e)$ edges of length at most 1 each. We will obtain a tree T' spanning a superset Y' of Y such that the length of each edge in T' is no more than 1. Note that the number of newly added Steiner points is

$$\begin{aligned} |Y'| - |Y| &= \sum_{e \in E(T)} \big(l'(e) - 1\big) \\ &= \sum_{e \in E(T)} l'(e) - \big|E(T)\big| \\ &\leqslant L - \big(|Y| - 1\big). \end{aligned} \tag{2}$$

Therefore the number of Steiner points in T' is

$$\begin{aligned} |Y'| - |X| &= |Y'| - |Y| + (|Y| - |X|) \\ &\leqslant L - (|Y| - 1) + (|Y| - |X|) \\ &= L - (|X| - 1) \\ &= B. \end{aligned} \tag{3}$$

This shows that T' is a solution to I'. To summarize, we have shown that *if the answer to I' is* NO *then the answer to I is also* NO. This completes our proof of the theorem. □

Given the number of Steiner points and a topology which specifies the edges in the final tree, a bottleneck tree under this given topology and edge-bound constraint can be computed in polynomial time [9]. Therefore Problem 2 belongs to the class NP. It follows from this remark and Theorem 3 that we have the following theorem.

Theorem 4. *Problem* 2 *is* NP*-complete.*

3. The MST heuristic

In this section, we will present a simple heuristic which will be called the *Minimum Spanning Tree (MST) Heuristic*. We will show that the MST heuristic for the STP-MSPBEL problem has a worst-case performance ratio of 5.

Let $P = \{p_1, p_2, \ldots, p_n\}$ be a set of n terminal points in the plane. Let R be the given edge bound. The MST heuristic first computes a minimum spanning tree over P. It then divides each edge in the tree into small pieces of length at most R. For each edge e in the minimum spanning tree, it inserts $\lceil l(e)/R \rceil - 1$ degree-2 Steiner points to subdivide edge e into equal pieces, where $l(e)$ is the (Euclidean) length of edge e. The tree so-obtained is clearly a feasible tree, denoted by T_A. Fig. 1 describes this algorithm.

Algorithm:	Minimum Spanning Tree Heuristic
Input:	A set P of n terminals, a given edge bound R.
Output:	A feasible tree T_A spanning P.
Step 1.	Compute a minimum spanning tree T for P.
Step 2.	Divide each edge in T into small pieces of length at most R using the minimum number of Steiner points.
Step 3.	Output the final tree as T_A.

Fig. 1. The MST heuristic algorithm for STP-MSPBEL.

For any (Steiner) tree T over P, let $\#(T)$ denote the number of Steiner points in T. In what follows, we will show that $\#(T_A)$ is not greater than five times the number of Steiner points in an optimal tree. This means that the MST heuristic has a performance ratio of at most 5, and thus it is actually an approximation algorithm. To facilitate the presentation, let $\mathcal{S}$ denote the set of optimal Steiner trees over P. Let T^* be a tree in $\mathcal{S}$ such that its length is minimum among all trees in $\mathcal{S}$. We will call such a tree a *shortest length optimal Steiner tree* for P.

Lemma 5. *Let T be a Steiner tree for the given terminal set P without Steiner points of degree more than two. Then, $\#(T_A) \leqslant \#(T)$.*

Proof. Removing Steiner points of degree two, T becomes a spanning tree for the given terminal set P. Similarly, ignoring Steiner points of degree two, T_A is a minimum spanning tree for P. From the property of a minimum spanning tree, we know that T_A can be obtained from T by a sequence of operations in each of which we remove an edge e in T and add an edge e_A in T_A such that

$$l(e) \geqslant l(e_A).$$

Therefore, the lemma holds. □

Lemma 6. *There exists a shortest length optimal Steiner tree for SMT-MSPBEL such that every Steiner point has degree at most five.*

Proof. Consider the tree T^*, a shortest length tree in $\mathcal{S}$. Note first that every edge in T^* is straight. It is also clear that every two adjacent edges form an angle of at least 60°, otherwise T^* could be shortened.

In the case that there are degree-6 Steiner points in T^*, we show that there exists another shortest length tree in $\mathcal{S}$ such that the number of degree-6 Steiner points in it is one less than that in T^*. It then follows that there exists a shortest length optimal tree in which every Steiner point has degree at most five.

Let q be a degree-6 Steiner point in T^*. From the property that every two adjacent edges are 60° apart, we know the six edges incident at q are of the same length (which is at most R). Since otherwise the tree T^* would be shortened. Let q' be a neighbor of q. We claim that q' has degree at most 4. This is true because we may "flip" two edges (two adjacent edges of edge $e[q, q']$) incident at q to q' without increasing the tree length and the number of Steiner points in the tree. This way, if q' has degree more than 4, then the angle condition would be violated. Now, since q' has degree at most 4, we flip one of the two edges from q to q'. The obtained tree is still a shortest length optimal tree. This finishes the proof. □

Now, let T^* be a shortest length optimal Steiner tree in which there is no degree-6 Steiner point. Decompose T^* into small subtrees at given terminal points of degree more than one such that in each subtree every given point is a leaf (this kind of subtrees are known as *full subtrees*). Let t be such a subtree. After duplicating all edges in t, we get an Eulerian loop for the set of vertices (including terminal and Steiner points) in t. Imagining that each edge in t is inflated a little to duplicate a copy, see Fig. 2, there is no self-crossing in the Eulerian loop. Note that every Steiner point occurs at most five times in the loop, and every terminal point occurs exactly once. Break some edge in this loop to get a tree t' (which is a path) connecting all the terminal points in t. The number of Steiner points in the tree (each copy is counted as one) is no more than five times the number of Steiner points in t. For each full subtree t of T, we may construct a tree t'. Pasting all of them together, we get a spanning tree for the terminal set P. By Lemma 5,

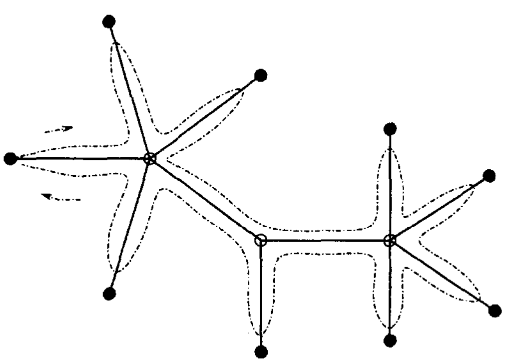

Fig. 2. Connecting terminals into a path.

$$5\#(T^*) \geqslant \#(T_A).$$

We have proved the following theorem:

Theorem 7. *The MST heuristic for the STP-MSPBEL problem has a performance ratio of at most* 5.

4. Conclusions

In this paper, we proved that the STP-MSPBEL problem is NP-complete. We have also shown that the simple MST heuristic algorithm has a worst-case performance ratio of at most 5. Since the STP-MSPBEL problem has important applications in VLSI design, WDM optimal networks and wireless communications, designing better approximation algorithms or providing better analysis on the performance ratio of the MST heuristic are important topics for future research.

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