ON REARRANGEABILITY OF MULTIRATE CLOS NETWORKS*

GUO-HUI LIN†, DING-ZHU DU‡, XIAO-DONG HU§, AND GUOLIANG XUE¶

Abstract. Chung and Ross [SIAM J. Comput., 20 (1991), pp. 726–736] conjectured that the multirate three-stage Clos network $C(n, 2n - 1, r)$ is rearrangeable in the general discrete bandwidth case; i.e., each connection has a weight chosen from a given finite set $\{p_1, p_2, \ldots, p_k\}$ where $1 \geq p_1 > p_2 > \cdots > p_k > 0$ and $p_i$ is an integer multiple of $p_k$, denoted by $p_k | p_i$, for $1 \leq i \leq k - 1$. In this paper, we prove that multirate three-stage Clos network $C(n, 2n - 1, r)$ is rearrangeable when each connection has a weight chosen from a given finite set $\{p_1, p_2, \ldots, p_k\}$ where $1 \geq p_1 > p_2 > \cdots > p_k > 1/2 \geq p_{h+1} > \cdots > p_k > 0$ and $p_{h+2} | p_{h+3} | p_{h+4} | \cdots | p_k | p_{k-1}$. We also prove that $C(n, 2n - 1, r)$ is two-rate rearrangeable and $C(n, \lfloor \frac{2n}{3}\rfloor, r)$ is three-rate rearrangeable.

Key words. rearrangeability, multirate Clos networks, minimization of the number of center switches

AMS subject classifications. 94A05, 05C70

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1. Introduction. The multirate interconnection network is a research topic in asynchronous transfer mode (ATM) networks with applications in computer networks, telecommunications, and the Internet. The symmetric three-stage Clos network $C(n, m, r)$ has been widely used in the design of telecommunication networks [1]. $C(n, m, r)$ consists of $r$ $n \times m$ crossbars (switches) in the first (or input) stage, $m$ $r \times r$ crossbars in the second (or central) stage, and $r$ $m \times n$ crossbars in the third (or output) stage. Every crossbar in the first stage has an outlet connected to an inlet of every crossbar in the third stage, and every crossbar in the second stage has an outlet connected to an inlet of every crossbar in the third stage (Fig. 1). There are $rn$ inlets in the first stage, called inputs, and totally $rn$ outlets in the third stage, called outputs. A connection (request, or call) in the network is a triple $(i, j, w)$ where $i$ is an input, $j$ is an output, while $w$ is the weight of this connection, and it represents the bandwidth required by the connection. A route is a path in the network joining an input crossbar (i.e., a crossbar in the first stage) to an output crossbar (i.e., a crossbar in the third stage) and a route $r$ realizes a connection $(i, j, w)$ if the input crossbar $i$ and output crossbar $j$ are connected by $r$ with capacity $w$.

Usually, one assumes that each link has unit capacity. Therefore, the weight of each connection is in the interval $[0, 1]$. A set of connections is compatible if, at every

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†Institute of Applied Mathematics, Chinese Academy of Sciences, Beijing 100080, People’s Republic of China. Current address: Department of Computer Science, The University of Vermont, Burlington, VT 05405 (guolin@emba.uvm.edu). The work of this author was supported in part by the National Natural Science Foundation of China.

‡Department of Computer Science, University of Minnesota, Minneapolis, MN 55455 (dzd@cs.umn.edu). The work of this author was supported in part by National Science Foundation grant CCR-9530306 and Grant-in-Aid Research of The University of Minnesota.

§Institute of Applied Mathematics, Chinese Academy of Sciences, Beijing 100080, People’s Republic of China (xdhu@amath3.amt.ac.cn). The work of this author was supported in part by the National Natural Science Foundation of China.

¶Department of Computer Science and Electrical Engineering, The University of Vermont, Burlington, VT 05405 (xue@emba.uvm.edu). The research of this author was supported in part by National Science Foundation grants ASC-9409285 and OSR-9350540.
input and output, the sum of weights of all connections is at most one. A request frame is a compatible set of connections. A configuration is a set of routes, and it is compatible if the total weight of routes passing through each link is at most one. A request frame is said to be realizable if there exists a compatible configuration which contains routes realizing all connections in the request frame. A multirate network is said to be (multirate) rearrangeable if every request frame is realizable.

A connection $c$ is said to be compatible with a request frame $\mathcal{F}$ if $\mathcal{F} \cup \{c\}$ is still compatible. A route $r$ is said to be compatible with a compatible configuration $\mathcal{C}$ if $\mathcal{C} \cup \{r\}$ is still compatible. A network is said to be strictly nonblocking if for every compatible configuration $\mathcal{C}$ realizing a request frame $\mathcal{F}$ and every connection $c$ compatible with $\mathcal{F}$, there exists a route $r$ such that $r$ realizes $c$ and $r$ is compatible with $\mathcal{C}$.

In circuit switching, all connections are assumed to have the same rate one. Namely, a network is said to be rearrangeable in circuit switching if every compatible request frame of connections with weight one is realizable, and it is well known that the symmetric three-stage Clos network $C(n, m, r)$ is rearrangeable in circuit switching if and only if $m \geq n$ [1]. Now, since multirate is involved, we may need more crossbars in the center stage to reach the rearrangeability. Chung and Ross [2] conjectured\footnote{After proving the result [2, Corollary 3] that a strictly nonblocking network for classical circuit switching is also rearrangeable if all connections have weight of either $b$ or 1, Chung and Ross [2] stated that “It would be of interest to show that Corollary 3 holds for the general discrete bandwidth case with $K$ distinct rates.” For an easy reference, we call it the Chung–Ross conjecture. In this paper, we consider this conjecture only for three-stage Clos networks.} that if a symmetric three-stage Clos network $C(n, m, r)$ is strictly nonblocking in circuit switching, then it is multirate rearrangeable in the discrete bandwidth case. That is, $C(n, 2n - 1, r)$ is multirate rearrangeable if each connection has weight chosen from a given finite set $\{p_1, p_2, \ldots, p_k\}$ where $1 \geq p_1 > p_2 > \cdots > p_k > 0$ and $p_i$ is an integer multiple of $p_k$, denoted by $p_k | p_i$, for $1 \leq i \leq k - 1$. They verified their conjecture when $k = 2$ and $\{p_1, p_2, \ldots, p_k\} = \{1, p\}$.

Du et al. [3] recently proved that $C(n, m, r)$ for $m \geq 41n/16$ is multirate rearrangeable in general.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{symmetric_three-stage_clos_network.png}
\caption{Symmetric three-stage Clos network.}
\end{figure}
In this paper, we prove that the symmetric three-stage Clos network $C(n, 2n - 1, r)$ is multirate rearrangeable when every connection has a weight chosen from a given finite set $\{p_1, p_2, \ldots, p_k\}$ where $1 \geq p_1 > p_2 > \cdots > p_h > 1/2 \geq p_{h+1} > \cdots > p_k > 0$ and $p_{h+2} | p_{h+1} | p_{h+3} | p_{h+2} \cdots | p_k | p_{k-1}$. We also prove that $C(n, 2n - 1, r)$ is two-rate rearrangeable and $C(n, \lceil \frac{2n}{r} \rceil, r)$ is three-rate rearrangeable.

2. Main results. In this section, we prove the following theorem.

Theorem 2.1. Symmetric three-stage Clos network $C(n, 2n - 1, r)$ is multirate rearrangeable when each connection has a weight chosen from a given finite set $\{p_1, p_2, \ldots, p_k\}$ where $p_{h+2} | p_{h+1} | p_{h+3} | p_{h+2} \cdots | p_k | p_{k-1}$, and $1 \geq p_1 > p_2 > \cdots > p_h > 1/2 \geq p_{h+1} > p_{h+2} > \cdots > p_k > 0$.

Proof. For $k - h = 0$, since $p_\ell > 1/2$ for all $\ell = 1, 2, \ldots, k$, each link contains at most one call. Thus, we can treat the network as that used in circuit switching. From the result about circuit switching, $C(n, 2n - 1, r)$ is nonblocking and hence rearrangeable. Next, we consider $k - h \geq 1$. Suppose $\alpha_\ell$ for $\ell = 1, 2, \ldots, h$ and $\beta$ are integers such that $p_\ell + \alpha_\ell p_k \leq 1 < p_\ell + (\alpha_\ell + 1)p_k$ and $\beta p_k \leq 1 < (\beta + 1)p_k$. First, route all calls with weights $p_1, p_2, \ldots, p_{k-1}$ on the $2n - 1$ center switches. By the induction assumption, it is possible. Now we route all calls with weight $p_k$ (or, say, $p_k$-call) in an arbitrary ordering. We will prove that we can always find a space compatible with previous routed calls. In fact, for contradiction, suppose there exists a $p_k$-call $(i, j, p_k)$ compatible with previous routed calls, but we cannot find space on a center switch to route this call $(i, j, p_k)$. Let $I$ be the input switch containing input $i$ and $J$ the output switch containing output $j$. Define the J-load (the J-load) of a center switch as the sum of weights of all calls from input switch $I$ to the center switch (from the center switch to output switch $J$). Then every center switch has its I-load or its J-load greater than $1 - p_k$. Note that there are $2n - 1$ center switches. Therefore, we can find either $n$ center switches that each has I-load greater than $1 - p_k$ or $n$ center switches that each has J-load greater than $1 - p_k$. Without loss of generality, we assume that the former occurs. This means that each of these $n$ center switches has I-load equal to either $p_\ell + \alpha_\ell p_k$ for some $\ell \in \{1, 2, \ldots, h\}$ or $\beta p_k$. Note that input switch $I$ has exactly $n$ input, and each input can contain at most one $p_\ell$-call for $\ell \in \{1, 2, \ldots, h\}$. Thus, every input in input switch $I$ has a load equal to either $p_\ell + \alpha_\ell p_k$ or $\beta p_k$. It follows that the call $(i, j, p_k)$ cannot exist, which is a contradiction. \[ \]

Melen and Turner [4] gave a routing algorithm CAP, and with CAP it can be showed that the multirate three-stage Clos network $C(n, 2n - 1, r)$ is rearrangeable when each connection has a weight at most $1/2$. The outline of the algorithm CAP is as follows.

Divide all calls in each input/output switch into groups of size $m$. Then algorithm CAP can arrange $m$ center switches, with flexible link capacity, to route all calls such that each center switch holds at most one call from every group in every input/output switch. Thus, if for each input/output switch the total weight of a set of calls chosen one from each group is never greater than one, then the line capacity can be restricted to be within one. Thus, algorithm CAP actually routes all calls with $m$ center switch in the model considered in this paper.

The following is a corollary of Theorem 2.1.

Corollary 2.2. The symmetric three-stage Clos network $C(n, 2n - 1, r)$ is multirate rearrangeable when every connection has a weight chosen from a given set $\{p_1, p_2\}$.

Proof. If $p_1 \leq 1/2$, then it follows from the result of Melen and Turner [4]. If
\[ p_1 > 1/2, \text{ it follows from Theorem 2.1.} \]

The next result is obtained with algorithm CAP, too.

**Lemma 2.3.** Symmetric three-stage Clos network \( C(n, 2n - 1, r) \) is multirate rearrangeable when each connection has a weight bigger than \( 1/3 \).

**Proof.** Since every call has weight bigger than \( 1/3 \), each input/output has at most two calls. Thus, each input/output switch has at most \( 2n \) calls. If it has fewer than \( 2n \) calls, we put them all into one group. If it has exactly \( 2n \) calls, then we place the one with the smallest weight among the \( 2n \) calls into one group and put the remaining \( 2n - 1 \) calls into another group. Note that when an input (output) switch has exactly \( 2n \) calls, each inlet (outlet) in this switch has exactly two calls. It follows from this fact that among weights of the \( 2n \) calls, the smallest one plus any other one cannot exceed one. This means that the total weight of any two calls, respectively, chosen from two groups is at most one. Therefore, algorithm CAP can route all calls with \( 2n - 1 \) center switches.

The following is a result about three rates.

**Theorem 2.4.** Symmetric three-stage Clos network \( C(n, \lceil \frac{2n}{3} \rceil, r) \) is multirate rearrangeable when every connection has a weight chosen from \( \{p_1, p_2, p_3\} \) where \( 1 \geq p_1 > p_2 > p_3 > 0 \).

**Proof.** By Theorem 2.1 and Lemma 2.3, we can assume that \( p_1 > 1/2 \geq p_2 \) and \( 1/3 \geq p_3 \). In the following, we consider two cases.

**Case 1.** \( p_1 > 1/2 \) and \( 1/3 \geq p_2 > p_3 \). It was proved in [3] that if all calls with weights bigger than \( 1/f \) (\( f \) is an integer) can be routed with \( c \geq 2n \) center switches, then at most \( \lceil \frac{(c - 2)/f - c + 2n}{n} \rceil \) additional center switches are needed to route all calls with weights at most \( 1/f \). Now we choose \( f = 6 \). If \( p_3 \leq 1/6 \), then all calls with weights bigger than \( 1/6 \) are \( p_1 \)- or \( p_2 \)-calls. They can be routed with \( 2n \) center switches since all calls with the same weight can be routed with \( n \) center switches. Therefore, the total number of center switches for routing all calls is at most \( 2n + \lceil (2n - 2)/6 \rceil < \lceil 7n/6 \rceil \). Hence, we may assume \( p_3 > 1/6 \).

Consider a bipartite graph \( G \) with two vertex sets, respectively, consisting of all inputs and all outputs and with all \( p_3 \)-calls and \( p_2 \)-calls as edges. Since \( p_2 > p_3 > 1/6 \), each vertex has degree at most five. By a lemma of de Werra [5], this graph can be decomposed into five edge-disjoint matchings. First, we can route calls in four matchings with \( \lceil 4n/3 \rceil \) center switches since \( 1/3 \geq p_2 > p_3 \). In fact, each matching for the bipartite graph \( G \) can be decomposed into \( n \) edge-disjoint matchings for the bipartite graph \( H \) between input switches and output switches, and hence four matchings for \( G \) give \( 4n \) matchings for \( H \). Moreover, each center switch can route three matchings for \( H \). Therefore, we need only \( \lceil 4n/3 \rceil \) center switches to route four matchings for \( G \).

Next, we consider calls in the fifth matching together with all \( p_1 \)-calls. We will route them with \( n \) center switches in the following way.

If \( p_1 + p_3 > 1 \), then there are at most \( n \) considered calls in each input/output switch. Therefore, \( n \) center switches are enough to route them by classic routing algorithm.

If \( p_1 + p_3 \leq 1 \) and \( p_1 + p_2 > 1 \), then each input/output switch has at most \( 2n \) calls in which there are at most \( n \) \( p_1 \)- or \( p_2 \)-calls. Thus, we can divide them into two groups of at most size \( n \) such that one group contains only \( p_2 \)-calls and the other one contains the remainders. Now, with the routing algorithm CAP of Melen and Turner [4], \( n \) center switches are enough to route all considered calls.

If \( p_1 + p_2 \leq 1 \), then each input/output switch has at most \( 2n \) calls in which there
are at most \( n \) \( p_1 \)-calls. Thus, we can divide them into two groups of size at most \( n \) such that one group contains only \( p_2 \)- or \( p_3 \)-calls and the other one contains the remainders. Now, the routing algorithm CAP of Melen and Turner [4] can also use \( n \) center switches to route all considered calls.

Case 2. \( p_1 > 1/2 \geq p_2 > 1/3 \geq p_3 \). Furthermore, if \( p_1 + p_2 \leq 1 \), then each input switch has at most \( 2n \) \( p_1 \)- or \( p_2 \)-calls. These at most \( 2n \) calls can be divided into two groups such that each group contains at most \( n \) calls and only one group contains \( p_1 \)-calls since each input switch has at most \( n \) \( p_1 \)-calls. Since \( p_1 + p_2 \leq 1 \), the sum of two elements chosen, respectively, from the two group is at most one. Thus, we can use \( n \) center switches to route all \( p_1 \)-calls and \( p_2 \)-calls by the routing algorithm CAP of Melen and Turner [4]. It is shown in [3] that a network which is rearrangeable for the classical circuit switching is multirate rearrangeable if all weights are in the interval \([b, 1/\lfloor 1/b \rfloor]\) for some \( 0 < b \leq 1 \). According to this result, \( n \) center switches are enough to route all \( p_3 \)-calls. Therefore, totally, \( 2n \) center switches are enough when \( p_1 + p_2 \leq 1 \). Next, we may also assume \( p_1 + p_2 > 1 \). An argument similar to that in the proof of Theorem 2.1 will be employed.

First, route all calls with weights \( p_1 \) and \( p_2 \) on the \( 2n - 1 \) center switches. By Corollary 2.2, it is possible. Now we route all calls with weight \( p_3 \) (or, say, \( p_3 \)-call) in an arbitrary ordering. We will prove that we can always find a space for a \( p_3 \)-call compatible with previous routed calls. In fact, for contradiction, suppose there exists a \( p_3 \)-call \((i, j, p_k)\) compatible with previous routed calls, but we cannot find space on a center switch to route this call \((i, j, p_k)\). Let \( I \) be the input switch containing input \( i \) and \( J \) the output switch containing output \( j \). Define the I-load (the J-load) of a center switch as the sum of weights of all calls from input switch \( I \) to the center switch (from the center switch to output switch \( J \)). Then every center switch has its I-load or its J-load greater than \( 1 - p_3 \). Note that there are \([7n/3]\) center switches. Therefore, we can find either \([7n/6]\) center switches that each has I-load greater than \( 1 - p_3 \) or \([7n/6]\) center switches that each has J-load greater than \( 1 - p_3 \). Without loss of generality, we assume that the former occurs. Since \( p_1 + p_2 > 1 \), every I-load greater than \( 1 - p_3 \) must be in the following forms:

\[
p_1 + k_1 p_3 \left( k_1 = \left\lfloor \frac{1 - p_1}{p_3} \right\rfloor \right),
\]

\[
p_2 + k_2 p_3 \left( k_2 = \left\lfloor \frac{1 - p_2}{p_3} \right\rfloor \right),
\]

\[
2p_2 + k_3 p_3 \left( k_3 = \left\lfloor \frac{1 - 2p_2}{p_3} \right\rfloor \right),
\]

\[
k_4 p_3 \left( k_4 = \left\lfloor \frac{1}{p_3} \right\rfloor \right).
\]

Suppose that there are \( x_1 \) center switches with I-load equal to \( p_1 + k_1 p_3 \), \( x_2 \) center switches with I-load equal to \( p_2 + k_2 p_3 \), \( x_3 \) center switches with I-load equal to \( 2p_2 + k_3 p_3 \), and \( x_4 \) center switches with I-load equal to \( k_4 p_3 \). Then we have

\[x_1 + x_2 + x_3 + x_4 \geq \lceil 7n/6 \rceil.\]
Without loss of generality, assume
\[ x_1 + x_2 + x_3 + x_4 = \lceil 7n/6 \rceil. \]
(If \( x_1 + x_2 + x_3 + x_4 > \lceil 7n/6 \rceil \), we delete some center switches from our consideration.)

Now we consider \( p_1 \)-calls and \( p_2 \)-calls only in the \( I \)-loads of \( \lceil 7n/6 \rceil \) center switches. Suppose that among \( n \) inputs of input switch \( I \), there are \( y_1 \) inputs each containing such a \( p_1 \)-call, \( y_2 \) ones each containing one such \( p_2 \)-call, \( y_3 \) ones each containing two such \( p_2 \)-calls, and \( y_4 \) containing only \( p_3 \)-calls. Note that the number of considered \( p_1 \)-calls and the number of considered \( p_2 \)-calls does not change and the total number of \( p_3 \)-calls in \( I \)-loads must be smaller than the maximum number of \( p_3 \)-calls which can be put in the inputs. Thus, we have

\[
\begin{align*}
x_1 &= y_1, \\
x_2 + 2x_3 &= y_2 + 2y_3, \\
k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4 &< k_1y_1 + k_2y_2 + k_3y_3 + k_4y_4.
\end{align*}
\]

That is,
\[
\begin{align*}
(x_1 - y_1) &= 0, \\
(x_2 - y_2) + 2(x_3 - y_3) &= 0, \\
k_1(x_1 - y_1) + k_2(x_2 - y_2) + k_3(x_3 - y_3) + k_4(x_4 - y_4) &= 0.
\end{align*}
\]

Therefore,
\[
\begin{align*}
(x_1 - y_1) + (x_2 - y_2) + (x_3 - y_3) + (x_4 - y_4) &< \left(1 - \frac{k_1}{k_3}\right)(x_1 - y_1) + \left(1 - \frac{k_2}{k_4}\right)(x_2 - y_2) + \left(1 - \frac{k_3}{k_4}\right)(x_3 - y_3) \\
&= \frac{2k_2 - k_3 - k_4}{k_4}(x_3 - y_3).
\end{align*}
\]

Note that
\[
0 \leq \left\lfloor \frac{2(1 - p_2)}{p_3} \right\rfloor - (k_3 + k_4) \leq 1,
\]
\[
0 \leq \left\lfloor \frac{2(1 - p_2)}{p_3} \right\rfloor - 2k_2 \leq 1.
\]

It follows that \( |2k_2 - k_3 - k_4| \leq 1 \). Now, we consider two subcases.

Subcase 2.1. \( p_3 \leq 1/4 \). In this case, we have \( k_4 \geq 4 \) and \( |x_3 - y_3| = |y_2 - x_2|/2 \leq \lceil 7n/6 \rceil/2 \). Thus, we have
\[
(x_1 - y_1) + (x_2 - y_2) + (x_3 - y_3) + (x_4 - y_4) < \lceil 7n/6 \rceil/8.
\]

Therefore,
\[
x_1 + x_2 + x_3 + x_4 < n + \lceil 7n/6 \rceil/8 \leq \lceil 7n/6 \rceil(6/7 + 1/8) < \lceil 7n/6 \rceil,
\]
which is a contradiction.

Subcase 2.2. $p_3 > 1/4$. In this case, we have $1 \leq k_2 \leq 2$, $0 \leq k_3 \leq 1$, and $k_4 = 3$. If $k_3 = 1$, then $2p_2 + p_3 \leq 1$. Then we must have $k_2 = 2$. Thus, $2k_2 - k_3 - k_4 = 0$. Therefore,

$$x_1 + x_2 + x_3 + x_4 < y_1 + y_2 + y_3 + y_4 = n,$$

which is a contradiction. (Note: The proof here shows that $2n - 1$ center switches are enough in this special situation.) Next, we assume $k_3 = 0$, i.e., $2p_2 + p_3 > 1$. Consider a bipartite graph with two vertex sets, respectively, consisting of all inputs and all outputs and with all $p_3$-calls and $p_2$-calls as edges. Since $p_2 > p_3 > 1/4$, each vertex has degree at most three. By a lemma of de Werra [5], this graph can be decomposed into three edge-disjoint matchings. Clearly, we can route two matchings with $n$ center switches since $1/2 \geq p_2 > p_3$. Put calls in the third matching together with all $p_1$-calls. We now consider $p_1$-calls and those calls in the third matching. If $p_1 + p_3 > 1$, then each input/output switch has at most $n$ considered calls. They can be routed with $n$ center switches. If $p_1 + p_3 \leq 1$, then each input/output switch has at most $2n$ such calls in which there exist at most $n$ $p_1$- or $p_2$-calls. Thus, we can route them with $n$ center switches by the routing algorithm CAP of Melen and Turner [4]. Thus, totally, $2n$ center switches are enough in this case. \[ \square \]

Note that in Subcase 2.2, only $2n$ center switches are required. Moreover, if $1 \geq p_1 > 1/2$ and $1/3 \geq p_2 > p_3 > 1/4$, then we can route all $p_1$-calls with $n$ center switches and route all $p_2$-calls and $p_3$-calls with $n$ center switches. Therefore, we have the following corollary.

**Corollary 2.5.** The symmetric three-stage Clos network $C(n, 2n, r)$ is multirate rearrangeable when every connection has a weight chosen from $\{p_1, p_2, p_3\}$ where $1 \geq p_1 > p_2 > p_3 > 1/4$.

3. **Discussion.** The conjecture of Chung and Ross [2] on rearrangeability of multirate Clos networks seems true not only in the discrete bandwidth case but also in arbitrary rates. This is equivalent to the following conjecture: consider any double stochastic square matrix of order $nr$. Divide it into $r^2$ blocks, each of which is an $n \times n$ submatrix. Now we color all cells of the matrix such that the total value in the same color and in the same block-row is at most one and the total value in the same color and in the same block-column is at most one. The conjecture says that $2n - 1$ colors are enough. In this paper, we proved it in several special cases. Finally, we would like to mention that by an argument similar to the proof of Lemma 2.3, we can also prove that the conjecture is true for $r = 2$.

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